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A New Theory of Elementary Matter. Part II: Electromagnetic and Inertial Manifestations

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Abstract

In accordance with the philosophical approach and its mathematical implications that were derived in Part I of this series (see p. 433), this paper deals explicitly with the manifestations of matter that are concerned with its electromagnetic and inertial properties. It is demonstrated that a logically and mathematically generalized version of electromagnetism emerges from extending the Faraday–Maxwell field approach, so as to fully unify these features of matter with the field description of matter itself. It is then shown how the most general expression of matter (according to the axioms of this theory), in terms of two-component spinor fields in a Riemannian space, leads to a derivation of the inertial properties of matter. The mass field so-derived (1) is a positive-definite function of the (global) coordinates—implying that gravitational forces can only be attractive; (2) approaches a discrete spectrum of values as the mutual coupling among the matter components of the closed system becomes arbitrarily weak; (3) predicts mass doublets in this approximation; and (4) approaches zero as the closed system becomes depleted of all other matter (in accordance with the Mach principle). It is also proven, as a consequence of the same field theory, that electromagnetic forces can be attractive or repulsive, depending on certain features of the geometrical fields of the Riemannian space.

1. *Electromagnetic Theory*

In view of the logical implication of the generalized Mach principle regarding the elementarity of the interaction rather than the free particle, there follows an interpretation of the Maxwell field equations that differs from the usual one. The interaction is described here in terms of the coupling of field variables which are associated with the components of a closed material system. Electromagnetic phenomena are expressible in terms of two types of field variables. One set relates to the field intensity that is conventionally associated with the electric and magnetic field variables. The other set relates to the 'source fields' that are conventionally identified with the charge density and its motion. According to the interpretation that is advocated here, Maxwell's equations are not more than a covariant prescription for determining one of these types of electromagnetic field variables in terms of the other. Thus, Maxwell's equations are interpreted as *an identity*. The actual physical observable is not described until the field variables describing one of the interacting components of the system are coupled to the field variables of the other interacting components of the closed system (in a relativistically invariant way).

With this interpretation, it follows that for each component of a closed system, there is a separate set of Maxwell's equations. It also follows that a matter component cannot be allowed to interact with its own force field —such a description would be redundant within this interpretation. Thus, for the formalism to be logically consistent with this feature, it must be so-constructed to leave out those mathematical terms that represent a quantity of matter acting on itself. The disappearance of these self-energy terms automatically removes divergent quantities which appear in the conventional quantum field theory and have their roots in the classical Lorentz theory of electrodynamics. Thus, these divergences do not have to be removed here, as it is done in the particle theories; they are not present from the outset!

Two other important deviations from the conventional theories follow from the proposed interpretation of the electromagnetic field equations. For it follows here that if the source fields should vanish, then the field intensities must also be identically zero, and vice versa. Thus, this approach rejects the homogeneous solutions of Maxwell's equations as unphysical; only the particular solutions are acceptable within the framework of this theory. The boundary conditions on the latter solutions of Maxwell's equations are automatically supplied by the feature of the coupled matter fields, which describe the remainder of the interacting system. These are the solutions of the coupled nonlinear matter field equations (to be derived in the next section) for the given *closed system* that is described. The latter property of the field solutions of this theory is a feature that any formalism must have if it is to incorporate the Mach principle.

It also follows from this interpretation of Maxwell's equations that the concept of the source-free radiation field and the photon (as an elementary particle) must be abandoned. This is because both of these follow from the homogeneous solutions of Maxwell's equations. The photon concept, which in particle theory is a quantum of the source-free radiation field, is replaced here with the *process* of energy-momentum transfer between systems of charged particles. As a consequence of this conclusion, it follows, for example, that an atom in an excited state would not decay spontaneously, with the emission of a free photon. The excited atom will only emit a signal (a quantity of energy-momentum transfer) if there is present in the system another atom to absorb this signal. Such a concept has already been discussed by other authors—it is commonly referred to as *action-at-a-distance* (or, more correctly, *delayed-action-at-a-distance*). The major difference between these previous *action-at-a-distance* theories and the present one is that the former are particle theories, in which one describes discrete charged matter trajectories in a $4n$-dimensional space (for an n-particle system), while this theory is a continuous field theory in which one has n coupled fields—all in the same four-dimensional space-time.

It should be remarked at this point that one motivation for the rejection of the photon, as a bona fide interacting particle, is the fact that the interpretation of most experimental effects which are conventionally attributed to the properties of free photons, can be equally explained in terms of the electromagnetic forces between charged ('quantized') matter that interacts over a large distance. This applies to the Compton effect, the photoelectric effect, bremsstrahlung, etc. It seems to this author that there are in fact only two sets of experimental data which are conventionally attributed to the properties of photons, when there is no matter around or where matter plays no role. One of these is the spectrum of blackbody radiation. The other is the data that is conventionally interpreted in terms of the annihilation of a particle–antiparticle pair, with the simultaneous creation of a pair of photons. It will be shown in Part IV (Sachs, 1971c) how both of these effects can be predicted from a particular bound state of the particle–antiparticle pair. Thus matter is not annihilated (or created) here, nor are photons created in order to explain the data. It is then concluded from these results that 'photons' need not be introduced as bona fide interacting particles to explain any phase of physical experimentation.

Finally, the interpretation of the electromagnetic field equations as an identity, and the rejection of the photon as an interacting particle, leads to the admissibility of a reduction of the vector representation of this theory to a lower dimensional form, if this would be possible within a covariant description of the theory. Such a possibility is indeed hinted at by the group theoretical feature, that the spinor representation is the most primitive one for any relativistically covariant formalism. Indeed, it has been shown (Sachs & Schwebel, 1962; Sachs, 1964a, 1971a) that a (first-rank) spinor form of the electromagnetic equations does contain all of the physical predictions of the conventional vector representation of the theory—in addition to extra predictions that have no counterpart in the vector theory.

Generalization to the Elementary Interaction Formalism

According to the interpretation of the electromagnetic field equations as identities, it follows (Sachs, 1964a) that these equations must be labelled, according to each of the interacting components of the closed system. In terms of the standard representation of the theory, we have

$$\begin{aligned} &\nabla \times \mathbf{E}^{(u)} + \partial^0 \mathbf{H}^{(u)} = 0 \qquad && \nabla . \mathbf{H}^{(u)} = 0 \\ &\nabla \times \mathbf{H}^{(u)} - \partial^0 \mathbf{E}^{(u)} = 4\pi \mathbf{j}^{(u)} \qquad && \nabla . \mathbf{E}^{(u)} = 4\pi\rho^{(u)} \end{aligned} \tag{1.1}$$
$$(c = 1)$$

where (u) stands for the uth interacting field component of the physical closed system.

The condition concluded above, that it would be logically inconsistent, within this theory, to allow a field component to interact with itself, implies that the conservation equations must be generalized in the following way:

$$\frac{1}{8\pi}\partial^0 \sum_{u\neq v}(\mathbf{E}^{(u)}.\mathbf{E}^{(v)}+\mathbf{H}^{(u)}.\mathbf{H}^{(v)})+\frac{1}{4\pi}\nabla.\sum_{u\neq v}(\mathbf{E}^{(u)}\times\mathbf{H}^{(v)})=-\sum_{u\neq v}\mathbf{E}^{(u)}.\mathbf{j}^{(v)}$$

$$\frac{1}{4\pi}\partial^0 \sum_{u\neq v}(\mathbf{E}^{(u)}\times\mathbf{H}^{(v)})=\sum_{u\neq v}(\rho^{(u)}\mathbf{E}^{(v)}+\mathbf{j}^{(u)}\times\mathbf{H}^{(v)}) \tag{1.1'}$$

where again, all fields are mapped in the same space-time coordinate system.

Clearly, as the number of interacting field components increases indefinitely, the corresponding macroscopic conservation equations (1.1′) start to lose sight of the underlying grid [that is labelled by (u) and (v)], and the resulting equations subsequently 'blur' into the standard form of the conservation equations in which these labels do not appear. In the latter form of the equations, the variables **E**, **H**, etc. are the sums of such variables over the indices (u) and (v). Thus the conservation equations (1.1′) do not differ in their predictions from the standard equations, when applied to low energy macroscopic phenomena (e.g., the application of Ohm's law, reception and transmission of radio signals, the scattering of 'electromagnetic radiation' from metallic or dielectric objects, etc.). On the other hand, the transmission of radio signals for example, would have to be viewed here in a different way. In the conventional case, one asserts that a radio antenna emits a signal at some time t. The signal then proceeds *on its own*, and at the later time, $t+R/c$, another antenna that is R cm away absorbs this signal. The later event is then said to be independent of the emitting antenna.

In contrast, the elementary interaction approach must, in principle, consider both the emitting antenna and the absorbing antenna *together* in terms of their *mutual influence*. Thus, one cannot reject the reaction of the emitting antenna to the absorbing antenna as is conventionally done. In practice, of course, the coupling between an individual radio set (or a city full of radio sets!) to the radio station transmitting antenna is certainly sufficiently weak to allow the description of the emitter and absorbers in terms of the solutions of uncoupled equations. This case corresponds to the *limit* of sufficiently small energy-momentum transfer, in which the predictions of the standard electromagnetic conservation equations and the generalized ones (1.1′) would merge. The latter situation corresponds to Faraday's original conception of the 'field' (for the 'emitter') and the uncoupled 'test charge' (the 'absorber').

A Spinor Formulation of Electromagnetic Theory (Sachs & Schwebel, 1962; Sachs, 1964a, 1971a)

The Maxwell formulation of electromagnetism was the first discovered law of physics that was found to be covariant with respect to the transformations of special relativity theory (the Poincaré group). The form of these field equations is in terms of a vector representation of the group.

However, it was found by Einstein & Mayer (1932) that the most primitive expression of any relativistically covariant formalism is in terms of two-dimensional complex, hermitian representations, whose basis functions are spinor variables. Thus, the insistence on using the four-dimensional representation of this group to describe electromagnetism would have to be based on physical assumptions that are in addition to the symmetry requirement of relativity theory. One such additional assumption that is made in elementary particle theory, is that a vector, massless particle—the photon—must be the primitive entity with which to describe electromagnetic coupling.

For the reasons discussed earlier, the theory advocated here does not accept the photon as a bona fide interacting particle. It then follows that a search for the lower-dimensional (spinor) representation of the electromagnetic field equations is in order. This will now be demonstrated by showing that Maxwell's field equations indeed factorize into a pair of uncoupled two-component spinor equations.

To demonstrate such a decomposition of Maxwell's equations, an initial identification must be made between the *real* components of the vector-tensor language, **E**, **H**, and the complex components of spinor variables. To do this, consider the complex vector whose spatial and temporal components are as follows:

$$G_k = (\mathbf{H} + i\mathbf{E})_k \qquad G_0 = 0$$

and let the structuring of the two-component spinor variables be guided by the correspondence between the features of the matrix representations of the Poincaré group—the hermitian components of a quaternion—and those of a 4-vector

$$\sigma_\mu x_\mu = \sigma_0 x_0 - \boldsymbol{\sigma}.\mathbf{r} \equiv \begin{pmatrix} x_0 - x_3 & -(x_1 - ix_2) \\ -(x_1 + ix_2) & x_0 + x_3 \end{pmatrix} \Leftrightarrow \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{1.2}$$

Considering this correspondence, an initial guess at the spinor structure of the electromagnetic equations in terms of the standard variables *in a given Lorentz frame* is as follows:

$$\begin{aligned} \varphi_1 &= \begin{pmatrix} G_3 \\ G_1 + iG_2 \end{pmatrix} \qquad & Y_1 &= -4\pi i \begin{pmatrix} \rho + j_3 \\ j_1 + ij_2 \end{pmatrix} \\ \varphi_2 &= \begin{pmatrix} G_1 - iG_2 \\ -G_3 \end{pmatrix} \qquad & Y_2 &= -4\pi i \begin{pmatrix} j_1 - ij_2 \\ \rho - j_3 \end{pmatrix} \end{aligned} \tag{1.3}$$

It is readily verified by direct substitution, with the quaternion differential operator defined as follows:

$$\sigma_\mu \partial^\mu = \sigma_0 \partial^0 - \boldsymbol{\sigma}.\boldsymbol{\nabla} \tag{1.4}$$

that the two 2-component spinor equations

$$\sigma_\mu \partial^\mu \varphi_\alpha = Y_\alpha \qquad (\alpha = 1, 2) \tag{1.5}$$

are in one-to-one correspondence with the standard form of Maxwell's equations (1.1).

At this stage, it is important to note that the identification (1.3) between the variables of the vector and spinor representations of the theory is only restricted to a given frame of reference. For, with the application of the transformations of the Poincaré group, the spinor variables transform in a way that there is no form-invariance with respect to the transformed variables **E** and **H**, i.e.

$$\varphi_\alpha(\mathbf{E}, \mathbf{H}) \not\xrightarrow{x \to x'} \varphi_\alpha'(\mathbf{E}', \mathbf{H}')$$

This is so because these relate to inequivalent representations of the Poincaré group.

But the *physical requirement* of the field theory does not require such form-invariant correspondence. It only requires a form-invariant correspondence in the invariants and conservation equations of the theory. This is because it is the latter, and not the field equations themselves, that are directly related to the observables. In view of the empirical validity of Maxwell's formalism in matching the data, it is required here that *all* of the invariants and conservation equations of the vector representation of the theory must correspond to at least some of those of the more general (spinor) formulation. Indeed, it will be shown below that the spinor formulation (1.5) of this theory contains invariants and conservation equations that are in one-to-one correspondence with all of those of the vector-tensor form of the theory. Thus, the spinor formulation predicts all of the physical consequences of the usual formulation. However, the spinor formulation contains additional invariants and conservation equations that have no counterpart in the usual formalism. It is then concluded that the 2-component spinor formulation (1.5) of electromagnetism is a true generalization of the vector–tensor formulation.

Invariants and Conservation Equations (Sachs & Schwebel, 1962)

The 2-component spinor equations (1.5) are relativistically covariant, if and only if

$$\varphi_\alpha(x) \xrightarrow{x \to x'} \varphi_\alpha'(x') = S\varphi_\alpha(x) \tag{1.6a}$$

$$Y_\alpha(x) \xrightarrow{x \to x'} Y_\alpha'(x') = (S^\dagger)^{-1} Y_\alpha(x) \tag{1.6b}$$

where $\{S\}$ are the solutions of the matrix equations

$$S^\dagger \sigma_\mu S = \alpha_\mu{}^\nu \sigma_\nu$$

and $\alpha_\mu{}^\nu = \partial^\nu x_\mu'$ are the elements of the corresponding group of continuous coordinate transformations of special relativity theory. It follows from the algebraic properties of spinor variables that the invariant metrics of the spinor formulation of electromagnetic theory are the terms

$$\varphi_1^{tr}\epsilon\varphi_2 = J_1 \qquad \text{and} \qquad Y_1^{tr}\epsilon Y_2 = J_2 \tag{1.7}$$

where 'tr' denotes a transposed spinor and

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is the Levi–Civita matrix.

If we now substitute into these invariants the identification with the conventional variables (1.3), then the following one-to-one correspondence between the invariants of the two formalisms results

$$J_1 \Leftrightarrow (E^2 - H^2) + 2i\mathbf{E}.\mathbf{H} \tag{1.8a}$$

$$J_2 \Leftrightarrow j^2 - \rho^2 \tag{1.8b}$$

The complex invariant J_1 of the spinor formalism then corresponds to two real invariants—which are the invariants of the standard formalism. The second invariant J_2 is a real number and corresponds to the modulus of the 4-vector j_μ.

In addition to the two invariants, J_1 and J_2, it follows from the transformation properties (1.6) that

$$I_{\alpha\beta} = \varphi_\alpha{}^\dagger Y_\beta \qquad (\alpha, \beta = 1, 2) \tag{1.9}$$

are four additional complex invariants (eight real invariants) of this formalism. These are to be compared with the one invariant, $j_\mu A^\mu$, which is used to describe the electromagnetic coupling in the vector representation. The extra invariants in equation (1.9) have no counterpart among the invariants of the standard formulation of electromagnetic theory.

In summary, we have established a one-to-one correspondence between some of the invariants of the spinor formulation of electromagnetic theory and all of those of the standard formalism. Additional invariants in the factorized spinor representation have no counterpart in the vector theory. It now remains to be shown that the generalization of this type carries over to the conservation equations.

If we multiply equation (1.5) on the left with the hermitian adjoint of φ_β,

$$\varphi_\beta{}^\dagger \sigma_\mu \partial^\mu \varphi_\alpha = \varphi_\beta{}^\dagger Y_\alpha$$

and take the hermitian adjoint of this equation, while interchanging the labels α and β, the following equation results:

$$\partial^\mu(\varphi_\beta{}^\dagger \sigma_\mu \varphi_\alpha) = (\varphi_\beta{}^\dagger Y_\alpha + Y_\beta{}^\dagger \varphi_\alpha) \tag{1.10}$$

These, in turn, correspond to eight real conservation equations, as contrasted with the four real conservation equations (1.1′), which have the equivalent form

$$\partial^\mu T_\mu{}^\nu = K^\nu \tag{1.11}$$

of the vector theory. Here, $T_\mu^{\ \nu}$ is the energy-momentum tensor for the electromagnetic field and

$$K^\nu = \{(\mathbf{E} + \mathbf{j} \times \mathbf{H}); -\mathbf{j}.\mathbf{E}\}$$

is the four-Lorentz force density.

It follows from the invariants (1.9) that the eight real force density terms on the right-hand side of equation (1.10) are separately scalar fields. Thus, equations (1.10) represents four complex *scalar* conservation equations, rather than the single vector conservation equation (1.11) of the standard formalism.

To exhibit the correspondence between the conservation equations of the standard formalism and some of those contained in equation (1.10), consider the sum of equation (1.10) with $\alpha = \beta = 1$ and equation (1.10) with $\alpha = \beta = 2$. It is readily verified, with the substitution of the identification (1.3) with the standard variables, that the sum of equations mentioned above yields the conservation of energy equation of the standard theory:

$$\frac{1}{8\pi}\partial^0(E^2 + H^2) + \frac{1}{4\pi}\nabla.(\mathbf{E} \times \mathbf{H}) = -\mathbf{E}.\mathbf{j}$$

It is also found that the conservation of momentum equations [in equation (1.1′)] follow from other sums and differences of the conservation equations (1.10).

The Lagrangian

The least action principle yields the factorized spinor equations (1.5) when the Lagrangian density has the form:

$$\mathscr{L}_M = ig_M \sum_{\alpha=1}^{2} (-1)^\alpha \varphi_\alpha^\dagger(\sigma_\mu \partial^\mu \varphi_\alpha - 2Y_\alpha) + \text{h.c.}$$

In this functional form, we have chosen to write the (otherwise) arbitrary set of two coefficients for each of the two spinor fields, φ_1 and φ_2 as $(-1)^\alpha$, because of a later application of the theory to electron-positron systems (Sachs, 1971c). With this choice, we will show that one obtains from the formalism a prediction of a particular bound state of the electron–positron pair, which can be associated with all of the observed properties associated with pair annihilation and creation.

When this Lagrangian density is varied with respect to $\varphi_1^\dagger$ and $\varphi_2^\dagger$ we obtain the two spinor equations (1.5). The constant g_M cancels out (as it appears on both sides of the equation). However, when this Lagrangian is varied with respect to the matter field variables that are implicit in Y_α, we obtain an interaction that corresponds to the term

$$2ig_M \sum_{\alpha=1}^{2} (-1)^\alpha \varphi_\alpha^\dagger Y_\alpha + \text{h.c.}$$

This interaction has no counterpart in the usual formulation of electromagnetic theory. It will be shown (Part IV) to yield an important contribution to the fine structure of hydrogen—it predicts a Lamb splitting that is in very good numerical agreement with the data.

Note that the field term that g_M multiplies has the dimension of energy density per length. Thus, the fundamental constant g_M must have the dimension of length. It will be found from the analysis of the hydrogen spectrum that this constant is of the order of 2×10^{-14} cm. We then see that a new fundamental constant (of this magnitude) necessarily enters the theory as a consequence of the factorization of the Maxwell formalism into a pair of uncoupled 2-component spinor field equations [equation (1.5)].

The next generalization of the theory—that would fully incorporate the Mach principle—implies that for each matter component of a closed system, there is a separate set of spinor equations

$$\sigma_\mu \partial^\mu \varphi_\alpha^{(u)}(x) = \Upsilon_\alpha^{(u)}(x) \tag{1.12}$$

with the corresponding set of conservation equations

$$\partial^\mu \sum_{u \neq v} (\varphi_\beta^{(u)\dagger} \sigma_\mu \varphi_\alpha^{(v)}) = \sum_{u \neq v} (\varphi_\beta^{(u)\dagger} \Upsilon_\alpha^{(v)} + \Upsilon_\beta^{(u)\dagger} \varphi_\alpha^{(v)}) \tag{1.13}$$

Using the method of Fourier transforms, it is readily verified (Sachs, 1971a) that the particular solutions of the electromagnetic equations (1.12)—the only solutions that are acceptable within the proposed theory—are the following for a point charge, e, at the origin, acting on a test charge at r

$$\varphi_1 = -(ie/r^3)\begin{pmatrix} x_3 \\ x_1 + ix_2 \end{pmatrix} \qquad \varphi_2 = (ie/r^3)\begin{pmatrix} x_1 - ix_2 \\ -x_3 \end{pmatrix} \tag{1.14}$$

These are the field solutions that replace the solutions

$$E_j = ex_j/r^3, \qquad H_j = 0 \qquad (j = 1, 2, 3)$$

of the conventional formalism, for a point charge at the origin, acting on a test charge at r.

Using this result, it was shown (Sachs, 1971a) that with the proper choice for the source field Υ_α of the test point charge, the predictions of this theory are in one-to-one correspondence with those of the conventional prediction of the Coulomb force between the point charges considered.

Even though the predictions agree in this particular application of the spinor representation of electromagnetism, with those of the vector representation of the theory, it is important to note that the two theories, in their general forms, are quite different. Where the approximations used above (e.g., a fixed point charge at a special place) would become inaccurate, differences would occur in the outcome of both theories. The reason is that, in general, the spinor formulation contains more predictions than the vector formulation. These differences, however, do not show up until one is forced to use a full relativistic treatment. Some of these effects will be derived in Part IV (Sachs, 1971c) that treats 'pair annihilation' and the fine structure of hydrogen.

The Electromagnetic 4-Potential

We will see in the section that follows that an interaction term must appear in the matter field equations which entails the coupling of the electromagnetic current density, for one interacting component of the system, to the electromagnetic 4-potential, for the other interacting components of the system. In terms of the spinor fields of the matter equations, the 4-current density has the usual form

$$j_\mu^{(u)} = e\bar{\psi}^{(u)}\gamma_\mu\psi^{(u)} \qquad \text{or} \qquad e\eta^{(u)\dagger}\sigma_\mu\eta^{(u)} \tag{1.15a}$$

The 4-potential solves the vector equation $\Box A_\mu^{(u)} = 4\pi j_\mu^{(u)}$ and

$$A_\mu^{(u)} = e\int j_\mu^{(u)}(x')\,S(x-x')\,d^4x' \tag{1.15b}$$

are the particular solutions of this equation, where $S(x-x')$ are the Green's functions for the field equation in $A_\mu^{(u)}$. Among the different possibilities for S, the elementary interaction approach requires that only the symmetrized form can be used. This is the function:

$$S(x-x') = \frac{1}{4\pi^3}\int \frac{d^4k}{k^\rho k_\rho}\exp\{i[k^\mu(x_\mu - x_\mu')]\} \tag{1.16}$$

$$= \frac{1}{2|\mathbf{r}-\mathbf{r}'|}\{\delta[(t-t') - |\mathbf{r}-\mathbf{r}'|] + \delta[(t-t') + |\mathbf{r}-\mathbf{r}'|]\}$$

which corresponds to an average of the retarded and advanced terms in the potential.

The retarded potential alone, is the usual one that is used in electrodynamics. It is consistent with the particle theory in which one considers one charged particle to emit a signal at some initial time, t_0, and the second charged particle to absorb this signal at the later time $t_0 + R/c$, if the particles are separated by R cm. With such a model, one does not consider that the advanced potential can play any role, since it seems to imply an effect that precedes its cause, thereby implying a violation of the principle of causality.

On the other hand, when 'emitter' and 'absorber' are only names which are assigned for convenience—when the overall description is covariant with respect to the interchange of all of the variables associated with these two components of an interacting system—then such a theory must select only that Green's function that yields a potential which is symmetric with respect to the advanced and retarded terms. As we have indicated earlier, the incorporation of the Mach principle in the field theory under study, requires such a symmetry between the emitter and the absorber and therefore uniquely chooses this Green's function in the description of $A_\mu^{(u)}$. This result will play an important role in the application of this theory to electrodynamics (Sachs, 1971b, c).

In this section we have concentrated on the form of the electromagnetic equations that is implied by the field theory of elementary interactions. In the following section, we will derive the equations that relate to the inertial properties of matter and its quantum mechanical nature.

2. *Theory of Inertia and the Matter Field Equations*

We have already discussed the *general* mathematical structure of a field theory of matter which must follow from the three axioms of this study It has been argued that the most primitive form of the field equations to describe matter must be in terms of at least two coupled non-linear spinor field equations. In the preceding section we discussed the form and the interpretation of the electromagnetic coupling that must eventually appear in the matter field equations.

In this section, the *explicit* mathematical structure of the matter field equations will be derived, in accordance with the three underlying axioms of this theory. It will be shown that, without inserting any mass parameter into the equations, a particular mapping of time-reversed 2-component spinor fields in a curved space-time automatically yields a positive-definite field in the place where the mass parameter is normally inserted in a Dirac-type wave equation. Further, the imposition of gauge invariance on these field equations yields the result that as a consequence of the structure of the space-time, a non-positive-definite function must appear in the matter field equations that has the form of a vector coupling potential. These results must, of course, persist in the local limit of the theory—the limit where the special relativistic form of the theory is a good mathematical approximation for the generally covariant equations. The implication of these derived features of the formalism is then that gravitational forces can have only one sign—they are either attractive or repulsive, *not both*, but that electromagnetic forces can be attractive or repulsive. In the former case, it only takes one observation to establish whether the gravitational force is attractive or repulsive (e.g. the earth–sun attraction)—thereby establishing the prediction that gravitational forces must always be attractive. These derived results are in accord with the experimental facts about gravitational and electromagnetic forces, and have never been derived by any other theory. Here they are consequences of fully exploiting the three axioms of this theory.

A further interesting consequence of this analysis is that it is in full accord with the interpretation of inertia according to the Mach principle. We will see that, in accordance with this principle, the inertial mass of any quantity of matter is a measure of its coupling with all of the other matter within a closed system—if all of the other matter in the system should tend to vanish, the mass of the remaining quantity of matter would correspondingly go to zero in an explicit way.

Finally, it will be shown that as a consequence of Axiom 3, applied to the structure of these equations, as the local domain is approached, the distribution of values of the inertial mass of matter in the microscopic domain approaches a discrete spectrum. It is important within this theory that the actual limit of a discrete spectrum cannot be reached—it can only be approached arbitrarily closely, corresponding to arbitrarily sharply peaked values for the mass spectrum. But in the actual limit—which in this analysis corresponds to a 'free particle'—the inertial mass values all go to zero. The 'free particle' states do not exist within this theory simply because of the implication of the Mach principle that the interaction between matter and matter can never be 'off', even though it can be arbitrarily weak. This consequence of the theory implies that the use of a mass parameter in the special relativistic form of the matter field equations (a form that approaches the form of the quantum mechanical equations) is in fact inserting an (averaged) field (in the curved space-time) into equations that are approximated by fields in a flat space-time.

The Appearance of Quaternions

Before presenting the derivation of the matter field equations and the inertial mass of matter, let us briefly discuss some of the basic features of quaternion analysis that will play an important role in the derivations to follow.

In the early part of the nineteenth century, W. R. Hamilton discovered the quaternion number field as a generalization of the field of complex numbers (Halberstam & Ingram, 1967). He found that the proper generalization of the basis elements $(1, i)$ of a complex number are the set of four basis elements that in fact are in one-to-one correspondence with the unit matrix, σ_0, and the three Pauli matrices, σ_k, i.e.

$$x + iy = z(1, i) \rightarrow \sigma_\mu x^\mu = Q(\sigma_0, \sigma_1, \sigma_2, \sigma_3) \tag{2.1}$$

Of course, Hamilton did not yet know about matrix algebra; it was not yet invented. But the properties that he discovered must be satisfied by the basis elements of the quaternions, in order that they satisfy the rules of an associative algebra, were in one-to-one correspondence with the multiplication and addition properties of the unit matrix and the three Pauli matrices.

Thus Hamilton discovered that, in effect, the three Pauli matrices are the proper generalization of the basis element i of a complex number. The latter were introduced about a century later by Pauli in order to describe the empirical facts about the angular momentum of a Schrödinger electron. The derivation of this effect, from first principles, followed from Dirac's discovery that to make the Schrödinger-type equation relativistically covariant, it became necessary to extend from the complex scalar field description to a complex multicomponent field description—the spinor. The general structure of Dirac's spinor field equations—in their most primitive form—will play an essential role in the analysis of this section.

It follows that the generalization of pairs of fields, in terms of the complex function, $u(x,y) + iv(x,y)$, are the quadruplets of fields, in terms of the quaternion function $\sigma_\mu A^\mu(x)$ [see equation (1.2)], where x represents the quadruplets of points in 4-space (x_0, x_1, x_2, x_3). Similarly, the generalization of the differential operator that underlies the calculus of complex functions is a quaternion differential operator,

$$d/dz \rightarrow \sigma_\mu \partial^\mu \tag{2.2}$$

The conjugate complex number, $\bar{z}(1,-i)$, is defined so that the product $z\bar{z}$ is invariant with respect to rotations in the two-dimensional coordinate system. Similarly, the conjugate quaternion, $\tilde{Q}(-\sigma_0;\ \sigma_k)$ (or $\bar{Q}(\sigma_0;\ -\sigma_k)$, where $Q\tilde{Q} = -Q\bar{Q}$) is defined so that $Q\tilde{Q}$ (or $Q\bar{Q}$) is invariant to rotations in the four-dimensional coordinate system. The conjugated quaternion $\tilde{Q}$ is equivalent to a time-reversal of Q, and $\bar{Q}$ is equivalent to a space-reflection of Q. These are physically equivalent representations for the conjugated quaternion.

It follows from the definition of the conjugated quaternion that the product of the quaternion first-order differential operator, $\sigma_\mu \partial^\mu$, and its conjugate operator, $\tilde{\sigma}_\mu \partial^\mu$, is an invariant. Writing this out, we find that

$$(\sigma_\mu \partial^\mu)(\tilde{\sigma}_\nu \partial^\nu) = (\partial^0)^2 - \nabla^2 \equiv \square \tag{2.3}$$

where $\square$ is called the D'Alembertian operator.

Spinor Field Equations

We have indicated in the preceding section that the most primitive irreducible representations of the Poincaré group are in terms of a quaternion number field. Thus the quaternion representation of the differential operator to appear in a relativistically covariant field equation is more primitive than any other form of differential operator. Indeed, we have seen above that the D'Alembertian operator factorizes into a product of a quaternion differential operator and its conjugate operator. This implies that the more basic differential equation would be the one governed by the (first-order) differential operator $\sigma_\mu \partial^\mu$, rather than the equation governed by the (second-order) operator $\square$. This, in fact, is the reason that the Klein–Gordon equation was found by Dirac to factorize into equations in spinor variables—the basis functions of the quaternion operators are the 2-component spinor functions. In terms of these variables,

$$(\square - \lambda^2)\eta = 0 \rightarrow \begin{cases} \sigma_\mu \partial^\mu \eta = -\lambda\chi & \text{(2.4a)} \\ \tilde{\sigma}_\mu \partial^\mu \chi = -\lambda\eta & \text{(2.4b)} \end{cases}$$

where η is a two component spinor field and $\chi = \epsilon\eta^*$ is the time reversal of η. ϵ is the Levi Civita matrix [equation (1.7)]. Had we chosen the space reflection conjugation $\bar{Q}$ rather than the time-reflection representation $\tilde{Q}$, equations (2.4) would then take the (physically equivalent) form in terms of the space-reflected spinor variables (ξ, ζ) solving the equations

$$(\square - \lambda^2)\xi = 0 \rightarrow \begin{cases} \sigma_\mu \partial^\mu \xi = -i\lambda\zeta & \text{(2.4a')} \\ \bar{\sigma}_\mu \partial^\mu \zeta = -i\lambda\xi & \text{(2.4b')} \end{cases}$$

As indicated in the preceding section (in terms of the spinor variables of the electromagnetic field) relativistic covariance of equations 2.4 is preserved if and only if

$$\eta(x) \xrightarrow{x \to x'} \eta'(x') = S\eta(x)$$

$$\chi(x) \xrightarrow{x \to x'} \chi'(x') = (S^{\dagger})^{-1}\chi(x)$$

where

$$S^{\dagger}\sigma_{\mu}S = (\partial x^{\mu}/\partial x^{\nu\prime})\sigma_{\nu}$$

It then follows that the invariants of this formalism are $\eta^{\text{tr}}\epsilon\eta$, $\chi^{\text{tr}}\epsilon\chi$ and $\eta^{\dagger}\chi$. The latter invariant can be expressed in the form $I_1 + I_2$, where

$$I_1 = \tfrac{1}{2}(\eta^{\dagger}\chi + \chi^{\dagger}\eta) \qquad (2.5a)$$

$$I_2 = \tfrac{1}{2}(\eta^{\dagger}\chi - \chi^{\dagger}\eta) \qquad (2.5b)$$

Since η and χ are the reflections of each other, it follows that I_1 is a scalar invariant with respect to reflections and that I_2 is a pseudoscalar—it is odd with respect to reflections. [Similar scalar and pseudoscalar invariants follow from the formalism (2.4′) with reflections here referring to the spatial coordinates rather than the time coordinate, the corresponding invariants will be called I_1' and I_2'.]

The usual bispinor form of Dirac's equation can be obtained from equations (2.4′), e.g., by combining these equations in such a way so as to yield a new equation in the 4-component function

$$\psi = \begin{pmatrix} \xi + \zeta \\ \xi - \zeta \end{pmatrix}$$

The equation in ψ takes the well-known form

$$(\gamma_{\mu}\partial^{\mu} + \lambda)\psi = 0 \qquad (2.6)$$

where γ_{μ}, the Dirac matrices, are built up from the Pauli matrices and the unit two-dimensional matrix (i.e. the quaternion basis elements) as follows:

$$\gamma_0 = -i\begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} \qquad \gamma_k = -i\begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$

According to the set of transformation properties of the field equation (2.6) that leave it relativistically covariant, there is only one scalar invariant

$$I_3 = \psi^{\dagger}\gamma_0\psi \qquad (2.7)$$

that is bilinear in ψ, but there is no pseudoscalar invariant. This can be seen by substituting the form of ψ in terms of ξ and ζ (above) into equation (2.7). It is found that

$$I_3 \propto (\xi^{\dagger}\zeta + \zeta^{\dagger}\xi) \Leftrightarrow I_1'$$

but that there is no counterpart in this formalism for the invariant I_2' of the 2-component spinor formalism.

We see, then, that the formal manipulation which led to the bispinor form of the equation (2.6) from the (more natural) form (2.4′) was equivalent to the process of eliminating the pseudoscalar invariant I_2'. In the language of quantum mechanics, this corresponds to extracting from a formalism that does not generally conserve parity, a part that does conserve parity, and eliminating the other part that would destroy reflection symmetry. Thus, this new (4-component bispinor) formalism is more symmetric than the theory of relativity would require, since this theory is based only on a symmetry depending on covariance with respect to *continuous coordinate transformations* between relatively moving coordinate frames.

The 2-component spinor form of the field equations does fully exploit the underlying symmetry of relativity theory, since it is covariant only with respect to the continuous transformations of the Poincaré group. The implication is that this formalism is more general than the bispinor formalism in the sense that it entails more invariants and conserved quantities to be associated with the observables. The bispinor formalism is, of course, useful whenever one wishes to describe experimental effects that are reflection symmetric. However, it has a disadvantage in that it masks other effects that are sensitive to the feature of the formalism that it is not really a theory that recognizes the reflection transformation in time or space. An example of this remark will appear later on, where it will be shown how the inertial mass of matter can be related to a mapping between time-reversed (or space-reflected) 2-component spinor variables in a curved space-time.

The extension of the field equations (2.4) to include interaction—an essential ingredient within the theme of this paper—is accomplished by generalizing the quaternion operator $\sigma_\mu \partial^\mu$ to the form $(\sigma_\mu \partial^\mu + \mathscr{I})$, where the quaternion $\mathscr{I}$ is geometrically a scalar (as is $\sigma_\mu \partial^\mu$) and is a functional that depends on all of the fields, associated with the rest of a closed system, that couple to the field η. Thus, for each interacting component spinor field $\eta^{(i)}$ of a closed system, the special relativistic form of the coupled equations that determines this field is

$$(\sigma_\mu \partial^\mu + \mathscr{I}_i)\eta^{(i)} = -\lambda_i \chi^{(i)} \tag{2.8a}$$

$$(\tilde{\sigma}_\mu \partial^\mu + \tilde{\mathscr{I}}_i)\chi^{(i)} = -\lambda_i \eta^{(i)} \tag{2.8b}$$

where

$$\mathscr{I}_i = \mathscr{I}_i(\eta^{(1)}, \eta^{(2)}, \ldots, \eta^{(i-1)}, \eta^{(i+1)}, \ldots)$$

is the interaction functional and

$$\tilde{\mathscr{I}} = \epsilon \mathscr{I}^* \epsilon$$

is the time reversal of this functional.

The field equations (2.8) are then a set of coupled nonlinear spinor equations that describe the closed system. According to the axioms of this theory, the conclusion was reached (in the preceding sections) that these equations must be covariant with respect to the interchange of the variables which are associated with the interacting components of the closed system ('observer' and 'observed'). This form of the field equations will be seen to lead to the eigenfunction form of the equations of quantum mechanics, in the asymptotic limit of sufficiently small energy-momentum transfer between the interacting components of the system.

The remainder of this section will be devoted to a *derivation* of the inertial manifestation of the interacting matter, as it is defined through the term λ in equations (2.8).

The insertion of λ $(=mc/\hbar)$ into equation (2.8) in the usual analysis assumes that the inertial mass of any quantity of free matter is one of its intrinsic properties. On the other hand, the interpretation of inertia, according to the Mach principle, requires that, most generally, λ should be a function of the entire closed system—that it has to do with the dynamical coupling between matter and matter. Incorporating this idea with the principle of general relativity—which implies that all of the manifestations of interacting matter can be represented by the relation between the points in space-time (i.e. geometry)—it follows that instead of inserting a mass parameter into equation (2.8)—to be adjusted later to the data—we should be able to derive this term as a function of the properties of space-time.

From the topological point of view, we may take the structure of the field equations (2.8) as representing a particular mapping between time-reversed spinor fields, with the mass parameter playing the role of a measure in this mapping. The procedure in the derivation to follow is then to omit this parameter from the equation and, instead, to generalize the quaternion operator $\sigma_\mu \partial^\mu$ by expressing it in a Riemannian space. The aim then is to find the particular mapping whose limiting form should give the special relativistic form (2.8) of the matter field equations, with the mass parameter appearing as a limiting property of a field that appears in the general form of the mapping. The assumption, then, is that the most primitive explicit expression for the inertial mass of interacting matter is in terms of the way in which it appears through these field variables in the generalization of equation (2.8).

Inertial Mass From Geometry

Our first task in this program is then to generalize the quaternion differential operator $\sigma_\mu \partial^\mu$ so as to express it covariantly in a Riemannian space-time. We start by redefining the invariant differential increment ds of a Riemannian space. Taking the hint from the fact that the irreducible representations of the Einstein group (and the Poincaré group) obey the algebra of quaternions, we will define ds as a quaternion (rather than a real number) as follows:

$$ds = q^\mu(x)\, dx_\mu \tag{2.9}$$

Here, the field $q^\mu(x)$ is *geometrically* a 4-vector. But each of the four components of this vector is, *algebraically*, a quaternion—thus depending on four real fields. The field q^μ is then a 16-component entity and ds is a sum of four quaternions—therefore it is a quaternion.

To construct a real number from a quaternion we multiply it by its conjugate $\widetilde{ds}$ and define the quantity to be the real number ds^2 of a Riemannian space, i.e.

$$ds\,\widetilde{ds} = -\tfrac{1}{2}(q^\mu \tilde{q}^\nu + q^\nu \tilde{q}^\mu)\,dx_\mu\,dx_\nu \equiv g^{\mu\nu}(x)\,dx_\mu\,dx_\nu \qquad (2.10)$$

The symmetrization on the left side of equation (2.10) results, of course, from the fact that dx_μ and dx_ν are commuting variables. The minus sign is chosen because of the way that these fields are normalized.

In the local limit, corresponding to very close separations in space-time, we require (for empirical reasons) that

$$q^\mu(x) \xrightarrow{\text{loc lim}} \sigma^\mu \qquad \tilde{q}^\mu(x) \xrightarrow{\text{loc lim}} \tilde{\sigma}^\mu$$

With this limit and the commutation properties of the Pauli matrices, it is readily verified that the corresponding limit of the squared increment ds^2 is the Lorentz metric, i.e.,

$$ds\,\widetilde{ds} \xrightarrow{\text{loc}} dx_0{}^2 - dr^2$$

i.e.

$$-\tfrac{1}{2}(q^\mu \tilde{q}^\nu + q^\nu \tilde{q}^\mu) \xrightarrow{\text{loc}} (1\,-1\,-1\,-1)\,\delta^{\mu\nu}$$

The field variables $q^\mu(x)$ solve equations that are a factorization of Einstein's field equations in $g^{\mu\nu}$. The procedure followed to derive these equations was to use the variational principle, expressing the Einstein Lagrangian R as a function of q^μ and $\tilde{q}^\mu$ (rather than $g^{\mu\nu}$) as the independent variational parameters. This results in equations that behave geometrically and algebraically as q^μ (rather than $g^{\mu\nu}$). These correspond to 16 relations at each space-time point. It is a more general description of the geometry than in Einstein's field formalism, since the latter entails only 10 relations at each space-time point.

It was shown (Sachs, 1968b, 1969b) that when the equations in q^μ are iterated once (with a conjugate solution $\tilde{q}^\nu$)—thereby forming a second-rank tensor representation of these equations—and when the iterated equations are expressed as the sum of a symmetric tensor part and an antisymmetric tensor part, the former is in one-to-one correspondence with Einstein's equations. Thus, the field $q^\mu(x)$ *contains* all of the physical predictions of Einstein's equations and therefore predicts the gravitational manifestations of interacting matter.

We have seen, then, that the generalization of the constant matrices σ_μ in the quaternion operator $\sigma_\mu \partial^\mu$ are the fields $q^\mu(x)$. The generalization of the ordinary derivatives ∂_μ are the covariant derivatives, defined as follows:

$$\eta_{;\mu} = \partial_\mu \eta + \Omega_\mu \eta \qquad (2.11)$$

where Ω_μ is the 'spin-affine connection'. This term arises (as in the tensor formulation) because of the fact that the ordinary derivatives of multi-component variables are not covariant entities in a curved space. An extra term must be added in order to force covariance on these operators. The explicit form of Ω_μ follows from the vanishing of the covariant derivatives of the quaternion fields, q_μ, and the fact that the latter is a vector, geometrically, and its algebraic properties define it as a second-rank spinor, of the type $\eta^* \otimes \eta$. The explicit form of the spin-affine connection is the following (Sachs, 1964a):

$$\Omega_\mu = \tfrac{1}{4}(\partial_\mu \tilde{q}^\rho + \Gamma^\rho_{\tau\mu}\tilde{q}^\tau)q_\rho = -\tfrac{1}{4}\tilde{q}_\rho(\partial_\mu q^\rho + \Gamma^\rho_{\tau\mu}q^\tau) \tag{2.12}$$

From this expression, we see that, geometrically, Ω_μ behaves covariantly as a 4-vector. On the other hand, it is not a covariant entity with respect to its spin degrees of freedom.

The generalization of the differential form $\sigma^\mu \partial_\mu \eta$ in a Riemannian space is then $q^\mu \eta_{;\mu}$. This entity is geometrically a scalar and algebraically a third-rank spinor. We shall set this form equal to $-\mathscr{I}\eta$ to yield the generally covariant spinor field equation

$$q^\mu \eta^{(i)}_{;\mu} = -\mathscr{I}_i \eta^{(i)} \tag{2.13}$$

It is important to note that our choice of the right-hand side of equation (2.13) to be (explicitly) linear in η is based on Axiom 3. According to this axiom—the assertion of a correspondence principle—equation (2.13) must approach the form of the eigenfunction equations of quantum mechanics in the local, low energy limit. This limit is approached when the functional

$$\mathscr{I}_i(\eta^{(1)}, \ldots, \eta^{(i-1)}, \eta^{(i+1)}, \ldots) \to \mathscr{I}(x)$$

This corresponds to the approximation in which one can describe the individual fields $\eta^{(i)}$ in an average background 'potential', analogous to the approximation in the nuclear many-body problem where one considers the nucleons in a nucleus, one at a time, in an average background field of all of the other nucleons. With this limit, the field equations (2.13) take the form of quantum mechanics. (This will be discussed in more detail in the succeeding sections.)

To complete the derivation of the particular mapping that is sought in order to identify the inertial mass field, consider the following hermitian and anti-hermitian matrices (Sachs, 1968a):

$$\Lambda_\pm = q^\mu \Omega_\mu \pm \text{h.c.} \tag{2.14}$$

It follows from the time-reversal properties of the quaternion fields that

$$\mathscr{T}q^\mu = \tilde{q}^\mu = \epsilon q^{\mu *} \epsilon \qquad \mathscr{T}\Omega_\mu = -\Omega_\mu{}^\dagger \qquad \mathscr{T}\Lambda_\pm = \pm\epsilon\Lambda_\pm{}^* \epsilon \tag{2.15}$$

The following identities are then readily verified

$$(\mathscr{T}\Lambda_\pm)\Lambda_\pm = \mp(\det \Lambda_\pm)\sigma_0 = \pm|\det \Lambda_\pm| \exp(i\delta)\sigma_0 \tag{2.16}$$

where the continuous set of values of the *real variables*, $\det \Lambda_\pm$ are mapped in a Riemannian space and $\delta = 0$ when $\det \Lambda_\pm < 0$ and $\delta = \pi$ when $\det \Lambda_\pm > 0$.

The matrix equation constructed from equation (2.16)

$$[(\mathscr{I}\Lambda_\pm)\Lambda_\pm]\eta = \pm(2\alpha_\pm)^2 \exp(i\delta)\eta \tag{2.16'}$$

may be factorized as follows:

$$\Lambda_+[\eta \exp(-i\delta/2)] = 2\alpha_+[\chi \exp(i\delta_+)] \tag{2.17a}$$

$$\mathscr{I}\Lambda_+(\chi \exp(i\delta_+)] = 2\alpha_+[\eta \exp(i\delta/2)] \tag{2.17b}$$

$$\Lambda_-[\eta \exp(-i\delta/2)] = 2i\alpha_-[\chi \exp(i\delta_-)] \tag{2.18a}$$

$$\mathscr{I}\Lambda_-[\chi \exp(i\delta_-)] = 2i\alpha_-[\eta \exp(i\delta/2)] \tag{2.18b}$$

where

$$(2\alpha_\pm)^2 = |\det \Lambda_\pm|$$

and the relationship between the time-reversed spinors is

$$\chi = \epsilon\eta^* \qquad \eta = -\epsilon\chi^*$$

The factorizations 2.17 and 2.18 of the matrix equation (2.16′) are unique only up to an arbitrary phase, relative to η and χ. Since these phases can be adjusted continuously without altering the form of the factorization, the equations may be grouped so that the relative phase between η and χ in (2.17) is the same as their relative phase in (2.18). In this way, equation (2.17a) (or equation (2.17b)) can be added to equation (2.18a) (or equation (2.18b)) without the need to explicitly specify the relative phase.

Combining the definition of the matrices $\Lambda_\pm$ (equation (2.14)) with the sum of equations (2.17a) and (2.18a), the following relationship between the time-reversed spinor variables is obtained:

$$q^\mu \Omega_\mu \eta = (\alpha_+ + i\alpha_-)\chi \equiv \lambda \exp(i\varphi)\chi \tag{2.19}$$

where

$$\lambda = \text{mod}(\alpha_+ + i\alpha_-) = \tfrac{1}{2}[|\det \Lambda_+| + |\det \Lambda_-|]^{1/2} \tag{2.20}$$

The phase angle of the complex function in equation (2.19) is

$$\varphi = \tan^{-1}\left(\frac{\alpha_-}{\alpha_+}\right) = \tan^{-1}\left|\frac{\det \Lambda_-}{\det \Lambda_+}\right|^{1/2} \tag{2.21}$$

The geometrical relationship that was sought between the time-reversed spinor fields in a Riemannian space is then given in equations (2.19), (2.20) and (2.21).

The time-reversed equation that accompanies eq. (2.19) is the following

$$-\tilde{q}^\mu \Omega_\mu{}^\dagger \chi = \lambda \exp(-i\varphi)\eta \tag{2.22}$$

The Dirac Equation in General Relativity

Inserting equation (2.19) into (2.13) (and suppressing the indices i) we have the generally covariant field equation

$$q^\mu \partial_\mu \eta + \lambda \exp(i\varphi)\chi = -\mathscr{I}\eta \tag{2.23a}$$

The time-reversed equation that accompanies this one is:

$$\tilde{q}^\mu \partial_\mu \chi + \lambda \exp(-i\varphi)\eta = -\tilde{\mathscr{I}}\chi \tag{2.23b}$$

If it were not for the phase factor, $\exp(\pm i\varphi)$, these would have the form of the 2-component spinor Dirac equation in a Riemannian space-time. To eliminate the phase, it is necessary to impose one further physical restriction. This is the requirement that (according to Axiom 3) there must be an equation of current conservation in the local limit of the theory. This equation takes the form

$$\partial_\mu(\eta^\dagger \sigma^\mu \eta) = 0 \tag{2.24}$$

in terms of the 2-component spinor formalism, or the form

$$\partial_\mu(\psi^\dagger \gamma^0 \gamma^\mu \psi) = 0 \tag{2.24'}$$

in terms of the bispinor formalism. The reason for this requirement has to do with the interpretation of the spinor field in terms of a weighting function.

The generally covariant extension of equation (2.24) is

$$(\eta^\dagger q^\mu \eta)_{;\mu} = 0$$

It is well known that the imposition of gauge invariance automatically leads to the current conservation equation. This corresponds to the following requirement on the 2-component spinor field equations:

$$\eta \to \eta \exp(-i\varphi/2), \qquad \chi \to \chi \exp(i\varphi/2) \Leftrightarrow \mathscr{I} \to \mathscr{I} + \frac{i}{2} q^\mu \partial_\mu \varphi \tag{2.25}$$

Using equation (2.25), it is readily verified (Sachs, 1968a) that the phase factors, $\exp(\pm i\varphi)$, in equation (2.23) can be transformed away—leaving the generally covariant spinor field equations

$$q^\mu \partial_\mu \eta + \lambda\chi = -\mathscr{I}\eta \tag{2.26a}$$

$$\tilde{q}^\mu \partial_\mu \chi + \lambda\eta = -\tilde{\mathscr{I}}\chi \tag{2.26b}$$

which have the Dirac form in terms of the 2-component spinor variables, approaching the linear eigenfunction form in the local non-relativistic limit.

We have seen that the form of the Dirac equation can be derived from a particular mapping between time-reversed spinor variables in a Riemannian space. The derived geometrical relationship is in terms of the field λ—*the* *modulus of a complex variable.* The latter field, which plays the role of the inertial mass parameter in these equations, is then a *positive-definite function.* Imposing now the equivalence between the gravitational and the inertial mass of interacting matter, it follows that gravitational forces can only have one sign—they are either attractive or repulsive in all cases. Since we observe that they are attractive in at least one case (e.g., the earth–sun attraction), the theory predicts that gravitational forces can only be attractive. This derived feature of gravitational forces is, of course, in agreement with all of the known experimental facts and has never been derived from first principles by other formulations of general relativity. It results here from a fundamental unification of the gravitational and inertial features of matter.

The Mass Spectrum

A second important feature of the derived mass field is that it is indeed a measure of the dynamical coupling between this matter, described by (η, χ), and all of the surrounding matter of a closed system. For, according to equation (2.20), λ depends on the curvature of space time, which in turn depends on all of the matter in the environment of any test matter that is described with the field λ. As the surrounding matter should diminish, the spin-affine connection correspondingly approaches a null matrix and λ tends to zero, in accordance with the general requirement of the Mach principle. It is important, however, that the sensitivity of the mass of matter to the detailed behavior of its environment (according to this analysis) is pronounced only in the microscopic domain, where the matter equations of the type (2.8) play an important role.

Combining equations 2.19 and 2.22, we have the relation

$$(-\tilde{q}^{\mu} \Omega_{\mu}{}^{\dagger})(q^{\nu} \Omega_{\nu}) \eta = \lambda^2 \eta \tag{2.27}$$

As indicated earlier, as $\mathscr{I} \to \mathscr{I}(x)$, the solutions of the spinor field equations (2.26) approach the form of the solutions of the linear eigenfunction equations of quantum mechanics. Thus, in this limit, the set of solutions of these equations approach the functions which are the *elements of a Hilbert space*, $\{\eta_s\}$. In this form, then, equation (2.27) can be re-written in terms of a spectrum of (averaged) mass values. Thus, this theory also predicts that in the microscopic domain, the distribution of values for the inertial mass of matter *approaches* a discrete spectrum:

$$\lambda_s{}^2 = \langle \eta_s | (-\tilde{q}^{\mu} \Omega_{\mu}{}^{\dagger})_{\lim} (q^{\nu} \Omega_{\nu})_{\lim} | \eta_s \rangle \tag{2.28}$$

This result is also in agreement with the general property of matter in the microscopic domain (and has not been predicted by other theories of elementary particles)—although at this stage, numerical results have not yet been obtained for the masses of elementary particles.

One further result emerges from this analysis. This is the feature that because the field operator in equation (2.27) does not commute with the operator in the equations (2.26) (because quaternions do not commute) the eigenvalues of the mass field, in terms of the solutions of equations (2.26), must be obtained by first diagonalizing the operator in (2.28). Since this is a two-dimensional matrix, it follows that for each solution η_s of the asymptotic form of the field equations, there are two mass values—*the spinor field variables predict mass doublets.* Each of these two 'particles' then are identical in all of their physical properties except for their masses. A well-known mass doublet of this type is the electron-muon pair. The quantitative substantiation of this particular result, however, awaits *numerical* predictions of the theory.

Electromagnetic Coupling

The application of gauge invariance on the spinor formalism led to the fact that a vector coupling term

$$\frac{i}{2}q^{\mu}\,\partial_{\mu}\varphi$$

must necessarily enter into these equations. If we now identify this term with the electromagnetic potential, it follows that if $\partial_{\mu}\varphi$ is a non-positive-definite function, then electromagnetic forces are either attractive or repulsive. According to the definition of the phase φ [equation (2.21)] the explicit form of $\partial_{\mu}\varphi$ is as follows:

$$\partial_{\mu}\varphi = \partial_{\mu}\tan^{-1}|\det\Lambda_{-}/\det\Lambda_{+}|^{1/2}$$
$$= \frac{1}{8\lambda^2}\{[|\det\Lambda_{+}|\,\partial_{\mu}|\det\Lambda_{-}| - |\det\Lambda_{-}|\,\partial_{\mu}|\det\Lambda_{+}|]/|\det\Lambda_{-}\det\Lambda_{+}|^{1/2}\} \tag{2.29}$$

It is clear that the right-hand side of this equation can be positive or negative, depending of the values of the metrical field variables that appear in the expression. Thus, it is concluded that if electromagnetic forces are defined most primitively through their appearance in the matter field equations, they can be either attractive or repulsive. This result, also in agreement with the experimental facts, has not been derived from first principles by other theories of matter.

Summing up, we have shown that: (1) The generally covariant form of the Dirac equation (which has the limiting Schrödinger form) can be derived from a particular mapping between time-reversed spinor variables in a Riemannian space. (2) The inertial mass of matter is derived from the metrical field. It is found to have the following features: (a) It is a positive-definite function—implying that gravitational forces can only be attractive (b) It has a spectrum of values in the microscopic domain that approaches discreteness in the asymptotic limit as the interaction between matter and matter becomes sufficiently weak. (c) The mass of a microscopic quantity of matter tends to zero as its massive environment correspondingly tends to zero—i.e., the mass of a 'free particle' is zero. (d) elementary particles occur in mass-doublets. (3) As a consequence of the requirement of incorporating a current conservation equation in the formalism, gauge invariance must be imposed. This in turn leads to the necessary appearance of a vector coupling term in the matter field equations. Identifying this term with electromagnetic forces, it is found, because of the dependence of this vector coupling term on the metrical field, that the electromagnetic force is non-positive-definite. This result then implies that electromagnetic forces can be attractive or repulsive—a result that is also in agreement with the experimental facts.

In the next article in this series, we will study the special relativistic form of the matter field equations that were derived here. This will be applied in particular to the case of electrodynamics for a closed system that approaches the description of a many-particle system in quantum mechanics.

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